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in the upper right, with the practical absence of analyses in the upper left and lower right quadrants, is very evident.

The chemical composition of igneous rocks is so complex and the number of their chemical constituents is so great; the conditions of differentiation and solidification are so varied and so complicated; and the knowledge of the application of the laws of physical chemistry to them is as yet so meager by reason of the lack of data, that some irregularities and apparent exceptions are to be expected. Furthermore there are other correlative relations which have not been dealt with here, and it is conceivable or indeed probable that one or more of these may under certain conditions, whether of composition, differentiation or solidification, supersede the one which is the subject of the present discussion without invalidating the truth of this.

It would be premature to enter here into a discussion of the causes of the correlation of these four elements. Any such would necessitate the consideration of the other correlations which have been detected. It may be said, however, that even though such concomitant variations may be due to similar solubility relations, or some such physico-chemical factor, yet that such similarities are themselves presumably due to certain intimate relationships between the elements which are commonly and somewhat loosely called affinities.

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[1] J. H. L. Vogt., *Zs. prakt. Geol.*, 1898, 326.

[2] J. F. Kemp., *Ore Deposits*, 34–37 (1900).

[3] L. de Launay, *La Science Géologique*, 637, 1905; *Gîtes Mineraux*, 1, 46–51 (1913).

[4] W. F. Hillebrand., *Bull. U. S. Geol. Surv.*, 305, 21 (1907).

[5] H. S. Washington., *Trans. Amer. Inst. Min. Eng.*, 1908, 749–767.

[6] H. S. Washington., *Prof. Paper U. S. Geol. Surv.*, 14, 1903.

THEOREM CONCERNING THE SINGULAR POINTS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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In earlier papers I have considered the effect of a linear transformation of dependent variables upon the solutions of ordinary linear differential equations in the vicinity of a singular point.

This type of transformation led me to the notion of *equivalence* which is fundamentally important for the classification of singular points.

Ordinary linear differential equations also preserve their form under an arbitrary transformation of the independent variable. I shall prove here that this second type has no additional significance for the purposes of classification since the solutions are carried over into the solutions of *equivalent* differential equations by such transformations.[1] The precise statement of the result is contained in the following

THEOREM. *Let* $Y(x)$ *be a matrix*[2] *of functions* $y_{ij}(x)$ $(i, j = 1, 2, \cdots, n)$, *each analytic for* $|x| \geqq r$, *with determinant* $Y(x)$ *not zero for* $|x| \geqq r$. *A necessary and sufficient condition that* $Y(x)$ *forms the matrix solution of a linear differential system*

$$\frac{dY}{dx} = A(x)\,Y,$$

where the elements $a_{ij}(x)$ *of* $A(x)$ *are analytic* $(q = 0)$ *or have a pole of at most the qth order* $(q > 0)$ *at* $x = \infty$, *is that, for every function* $\varphi(x)$ *of the form*

$$\varphi(x) = x + l\,x^{-q} + m\,x^{-q-1} + \cdots,$$

the matrix $M(x)$ *defined by the matrix equation*

$$Y(\varphi(x)) = M(x)\,Y(x)$$

is composed of elements $m_{ij}(x)$ *analytic at* $x = \infty$.

Let us prove first that the condition is *necessary.*

The function $M(x)$ defined for any choice of $\varphi(x)$ is single-valued as well as analytic for $|x|$ sufficiently large. In fact, when x makes a circuit of $x = \infty$, $Y(x)$ is altered to $Y(x)C$ where C is a matrix of constants of determinant not zero. But since $\varphi(x) - x$ remains finite as x becomes infinite, $\varphi(x)$ will make essentially the same circuit if $|x|$ be sufficiently great. Thus $Y(\varphi(x))$ changes to $Y(\varphi(x))C$, and $M(x)$ is unaltered in value.

Now the expression for $M(x)$

$$Y(x_0)\,Y^{-1}(x_m) \qquad\qquad (x_0 = \varphi(x);\; x_m = x)$$

may also be written as a matrix product

$$[Y(x_0)\,Y^{-1}(x_1)]\,[Y(x_1)\,Y^{-1}(x_2)]\cdots[Y(x_{m-1})\,Y^{-1}(x_m)].$$

Here we will assume that the values $x_0, x_1, x_2, \ldots, x_m$ of x are equally spaced, each after the first differing from its predecessor by $\Delta x = (x_1 - x_m)/m$.

From the differential equation itself, it is clear that

$$Y(x_i) - Y(x_{i+1}) = \Delta x[A(x_i) + E_i]\,Y(x_{i+1}) \qquad (i = 0 \text{ zero}, 1, \cdots, m-1)$$

where E_i will have arbitrarily small elements if Δx is small enough.

For $|x|$ taken large enough, it is also clear that we will have an equality of the form

$$|a_{ij}(x)| < a|x^q| \qquad (i, j, = 1, 2, \cdots, n)$$

for the elements of $A(x)$. Hence if I denotes the matrix with elements $\delta_{ij}(\delta_{ij} = 0,\ i \neq j;\ \delta_{ii} = 1)$, and if A denotes the matrix whose elements are all equal to the positive constant a, we find that the elements of the matrix

$$Y(x_i)\, Y^{-1}(x_{i+1}) \quad \text{or} \quad I + \Delta x(A(x_i) + E_i)$$

are less in absolute value than the corresponding elements of the matrix

$$I + 2|\Delta x| A |x_i|^q.$$

Also on account of the form of $\varphi(x)$ we have for $|x|$ large enough

$$|x_0 - x_m| \leqq k|x_i|^{-q}, \quad (k \text{ independent of } i \text{ or } x).$$

and thence

$$|\Delta x| \leqq \frac{k}{m}|x_i|^{-q}.$$

Substituting this value of Δx above, we conclude that the elements of $Y(x_i)Y^{-1}(x_{i+1})$ are less in absolute value than the corresponding elements of

$$I + \frac{2k}{m}A.$$

Thus, by the product formula, $M(x)$ will have elements whose absolute value is less than that of the corresponding elements of

$$\left(I + \frac{2k}{m}A\right)^m$$

whose elements are obviously limited in absolute value for all values of m.

Consequently the elements of $M(x)$ are limited in absolute value for $|x|$ large enough, and are necessarily analytic at infinity.

Let us show now that the stated condition is *sufficient*.

To this end we will consider the family of functions

$$\varphi(x) = x + \rho x^{-q}$$

where ρ is a parameter.

Defining $M(x,\rho)$ by the equation

$$Y(x + \rho x^{-q}) = M(x, \rho)\, Y(x),$$

we observe that $M(x,\rho)$ is made up of elements single-valued and analytic in x and ρ if

$$|x| \geqq 1 + r, \qquad |\rho| \leqq 1,$$

inasmuch as in this event

$$|x + \rho x^{-q}| \geqq |x| - |x^{-q}| \geqq |x| - 1 \geqq r$$

Moreover, in view of the stated condition, the elements $m_{ij}(x,\rho)$ of $M(x,\rho)$ are analytic in x at $x = \infty$.

Consider now the integral formula

$$\frac{1}{x} m_{ij}(x, \rho) = \frac{-1}{2\pi\sqrt{-1}} \int \frac{m_{ij}(\xi, \rho)\, d\xi}{\xi(\xi - x)} \qquad (|x| > 1 + r)$$

where the integration is performed around the fixed circle C: $|\xi| = 1 + r$ in a positive sense. This reduces to the Cauchy integral formula by the substitution $x = 1/x'$, $\xi = 1/\xi'$, which holds since $m(x,\rho)$ is analytic for $x \geqq 1 + r$ and at $x = \infty$.

We now perceive from the form of the integrand that $m_{ij}(x,\rho)$ must be analytic in x *and* ρ for $x = \infty$.

Differentiating the equation of definition for $M(x,\rho)$ with respect to ρ we obtain

$$x^{-q} \frac{d}{dx} Y(x + \rho x^{-q}) = \frac{d}{d\rho} M(x, \rho)\, Y(x).$$

The elements of $dM(x,\rho)/d\rho$ are also analytic in x and ρ at $x = \infty$. Letting ρ approach zero we obtain, for $|x| > 1 + r$,

$$x^{-q} \frac{dY(x)}{dx} = \frac{d}{d\rho} M(x, 0)\, Y(x)$$

which establishes at once that $Y(x)$ is the solution of a differential system of the desired form.

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[1] This result was partly known to me in 1908. For a special case, see ***Trans. Amer. Math. Soc.***, **14** 462–476 (1913), in particular pp. 475–476.

[2] For the elements of the theory of matrices here used see Schlesinger, ***Vorlesungen über*** *linearen Differentialgleichungen*, pp. 18–19.